22 Brief Introduction to Green's Functions: PDEs

In a previous section we discussed Laplace's equation in the disk with Dirichlet boundary conditions, namely

Example:

$$\begin{cases} \nabla^2 u = 0 \ , & \text{in } \Omega := \{(r,\theta) : 0 \le r < a \ , \ 0 \le \theta < 2\pi\} \\ u(a,\theta) = f(\theta) \ , & \text{for } 0 \le \theta < 2\pi, \ f \text{ is continuous} \end{cases}$$

and derived Poisson's formula for the solution,

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\psi)\, d\psi}{a^2 - 2ar\cos(\theta - \psi) + r^2} \ .$$

If we define $G(r,\psi) := \frac{1}{2\pi}\frac{a^2-r^2}{a^2-2ar\cos(\psi)+r^2}$, then we can rewrite u as

$$u(r,\theta) = \int_0^{2\pi} G(r,\theta-\psi) f(\psi)\, d\psi \ . \tag{1}$$

Thus, the form of solution is an integral, actually a convolution integral, with kernel $G(\cdot,\cdot)$ being considered the *Green's function* for the problem.

Example: For the IVP for the heat equation we found that

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} f(y)\, dy = \int_{-\infty}^{\infty} S(x-y,t) f(y)\, dy \ ,$$

which again a convolution integral for the solution. Sometimes $S(x,t)$, which we called the fundamental solution to the heat equation, is also called the Green's function for the heat equation.

Example: Consider now the Poisson problem

$$\begin{cases} \nabla^2 u = -f(x,y) & \text{in } \Omega = \{(x,y) : 0 < x < \pi, 0 < y < \pi\} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

For the homogeneous problem we have the eigenfunctions $\{\sin(nx)\}_{n\ge1}$ and from the Fourier transform idea, $u(x,y) = \sum_{n=1}^{\infty} b_n(y)\sin(nx)$, $f(x,y) = \sum_{n=1}^{\infty} f_n(y)\sin(nx)$, with

$$b_n(y) = \frac{2}{\pi}\int_0^{\pi} u(x',y)\sin(nx')dx' \ \text{ and } \ f_n(y) = \frac{2}{\pi}\int_0^{\pi} f(x',y)\sin(nx')dx' \ ,$$

and upon substitution of the series into the equation, we obtain

$$\frac{d^2 b_n}{dy^2} - n^2 b_n = -f_n(y) \quad 0 < y < \pi \ , \ b_n(0) = b_n(\pi) = 0 \ .$$

Thus,

$$b_n(y) = \frac{1}{n\sinh(n\pi)} \left\{ \int_0^y \sinh(n(\pi - y))\sinh(nz) f_n(z) dz \right.$$
$$\left. + \int_y^\pi \sinh(ny)\sinh(n(\pi - z)) f_n(z) dz \right\}$$

(For details, see Appendix at end of this section)

Therefore, in the Fourier sine series for $u(x, y)$ above, substitute the definition of $f_n(y)$ and interchange integration and differentiation to obtain

$$u(x, y) = \int_0^\pi \int_0^\pi G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \tag{2}$$

where

$$G(x, y; \xi, \eta) = \begin{cases} \sum_{n=1}^\infty \frac{2\sinh(n(\pi-y))\sinh(n\eta)}{n\pi\sinh(n\pi)} \sin(nx)\sin(n\xi) & \eta \le y \\ \text{same series but with the roles of } \eta \text{ and } y \text{ switched} & y \le \eta \end{cases}$$

Remark: This series can be expressed in closed form only in terms of elliptic functions, a calculation we will not pursue here. Still, the form of (2) leads one to think of this $G(x, y; \xi, \eta)$ as a Green's function for this Poisson problem.

Boundary value problems above have solutions that end up being expressed in terms of integrals whose integrands are either the boundary data or source functions times a kernel function we will call Green's function, G. The question arises whether such a Green's function and solution representation of a PDE in terms of an integral can be derived more directly. This question is motivated from ODE boundary value problems and associated Green's functions. We go through the construction of Green's functions for the solution of boundary value problems for the ODE case in Appendix J.

Exercise: Consider the Poisson problem defined on a square:

$$\begin{cases} \nabla^2 u = -1 & \Omega = \{(x,y):\ 0 < x, y < \pi\} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Let $v(x) = \frac{x}{2}(\pi - x)$ and note that $v(0) = v(\pi) = 0$ and $\nabla^2 v = -1$. Set solution to the above problem $u(x,y) = v(x) + w(x,y)$ and show that w satisfies $\nabla^2 w = 0$ in Ω, $w(0,y) = w(\pi,y) = 0$, $0 < y < \pi$, and $w(x,0) = w(x,\pi) = -\frac{1}{2}x(\pi - x)$, $0 < x < \pi$. This is a standard Laplace's equation problem, so solve it via separation of variables method.

Exercise: First, in the cartesian coordinate system, show the following:

1. Fix the point (ξ,η) in the plane and show that $G = G(x, y : \xi, \eta) = \frac{1}{2\pi}\ln(\frac{1}{R})$, where R is the distance between (x,y) and (ξ,η), i.e. $R = \sqrt{(x-\xi)^2 + (y-\eta)^2}$, is a solution to the problem $\nabla^2 u = u_{xx} + u_{yy} = 0$ in the plane, except at $(x,y) = (\xi,\eta)$.

2. Similarly, in 3-space, fix $\mathbf{x}' = (\xi,\eta,\zeta) \in \mathbb{R}^3$, and let $\rho = |\mathbf{x} - \mathbf{x}'|$ be the distance between $\mathbf{x}'$ and $\mathbf{x} = (x,y,z)$, and show that $G(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{1}{\rho}$ is a solution in $\mathbb{R}^3 \setminus \{\mathbf{x}'\}$ to $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$.

The $G's$ in the above exercise are the **free-space Green's functions** for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. But in bounded domains Ω where we want to solve the problem $\nabla^2 u = -f(\mathbf{x})$, $\mathbf{x} \in \Omega$, $u = 0$ on $\partial\Omega$, and be able to write the solution as $u(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')d\mathbf{x}'$, we need $G = 0$ on $\partial\Omega$. Therefore, we want G, the Green's function associated with the domain Ω, to have the form $G = \frac{1}{2\pi}\ln(\frac{1}{R}) + g(\mathbf{x},\mathbf{x}')$ in the 2D case, and $G = -\frac{1}{4\pi}\frac{1}{\rho} + g(\mathbf{x},\mathbf{x}')$ in the 3D case. Thus, g must be found so that G vanishes on the boundary $\partial\Omega$, and g is harmonic in Ω. This is difficult to do in general, but in some simpler cases it can be done via a **reflection principle**. (In 2D, there are also complex variable methods to find Green's functions, but we will not delve into that methodology in these Notes.)

The reflection principle is motivated by some physics experiments where a potential field due to positive and negative charges placed at a given distance apart gives rise to curves of constant potential; so there exists a boundary of points between the two charges where there is zero potential. As a test problem consider the Poisson problem on the upper half-plane, namely

$$\begin{cases} \nabla^2 u = -f(x,y) & \text{in } \Omega = \{(x,y); y > 0\} \\ u = 0 & \text{for } (x,0) \in \partial\Omega \end{cases}$$

Fix $\mathbf{x}' = (\xi, \eta) \in \Omega$ (maybe think of it as the location of a point positive charge). Then, can we locate a "negative" point charge outside Ω and use this to construct the appropriate g? Yes, let $\mathbf{x}'' = (\xi, -\eta)$. Then let

$$G(\mathbf{x}, \mathbf{x}') = G(x, y, \xi, \eta) =$$

$$\frac{1}{2\pi} \ln\left(\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\right) - \frac{1}{2\pi} \ln\left(\frac{1}{\sqrt{(x-\xi)^2 + (y+\eta)^2}}\right) =$$

$$\frac{1}{4\pi} \ln\left\{ \frac{(x-\xi)^2 + (y+\eta)^2}{(x-\xi)^2 + (y-\eta)^2} \right\} .$$

It is clear that $G = 0$ for $(x,y) = (x,0)$, and is harmonic for $(x,y) \neq (\xi,\eta)$. Thus,

$$u(x,y) = \int_0^\infty \int_{-\infty}^\infty G(x,y,\xi,\eta) f(\xi,\eta) \, d\xi \, d\eta \ .$$

Observe that we first fix a point $(\xi,\eta) = \mathbf{x}' \in \Omega$ and then construct G as a function of $\mathbf{x} = (x,y) \in \Omega$ such that G satisfies the boundary condition. We do this by using a version of $\ln(1/R)$ (or $1/\rho$ in the 3D case) to preserve the harmonic character of G. In the above example, the reflection point $\mathbf{x}''$ is outside $\Omega =$ upper half-plane, so $\ln(\frac{1}{|\mathbf{x}-\mathbf{x}''|})$ is harmonic everywhere in Ω. Of course, when we write out $u(x,y)$, then for each (fixed) $(x,y) \in \Omega$, G is integrated against the problem's non-homogeneity with respect to $\mathbf{x}' = (\xi,\eta)$.

A slight variation of this is to rotate and translate the above problem so that $\Omega = \{(x,y) \ : \ x < 1, \ |y| < \infty\}$. Then

Example 2: Again fix $\mathbf{x}' = (\xi,\eta) \in \Omega$ (so $\xi < 1$). We need to reflect about the boundary $\{x = 1\}$ a point $\mathbf{x}'' = (\xi'', \eta)$ so that $\ln(1/|\mathbf{x}-\mathbf{x}'|) - \ln(1/|\mathbf{x}-\mathbf{x}''|) =$

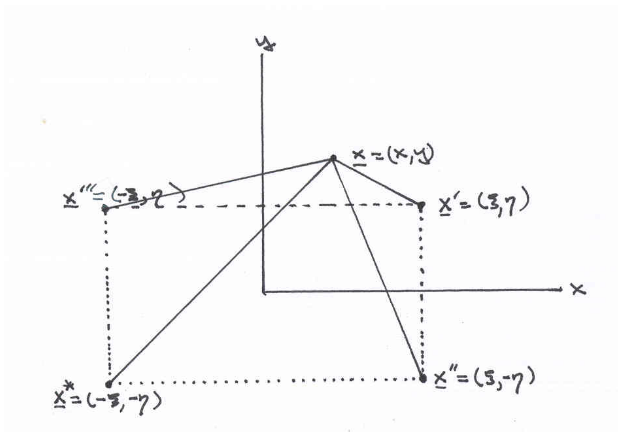

Figure 1: For Example 3, with two boundary pieces, we have to have three reflected points.

0 when $\mathbf{x} = (1, y)$. Hence, $\xi'' - 1 = 1 - \xi$, or $\xi'' = -\xi + 2$ and

$$G(\mathbf{x}, \mathbf{x}') = G(x, y, \xi, \eta) =$$

$$\frac{1}{2\pi}\ln(\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}) - \frac{1}{2\pi}\ln(\frac{1}{\sqrt{(x+\xi-2)^2 + (y-\eta)^2}}) =$$

$$\frac{1}{4\pi}\ln\left\{\frac{(x+\xi-2)^2 + (y-\eta)^2}{(x-\xi)^2 + (y-\eta)^2}\right\}\ .$$

Example 3: Consider the quarter-plane problem, that is

$$\begin{cases} \nabla^2 u = -f(\mathbf{x}) & \Omega = \{(x,y):\ x > 0, y > 0\} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

To have $G_{|y=0} = 0$ we do what we did in the first example, namely reflect about the boundary $\{y = 0\}$ to obtain

$$\frac{1}{2\pi}\ln(\frac{1}{|\mathbf{x} - \mathbf{x}'|}) - \frac{1}{2\pi}\ln(\frac{1}{|\mathbf{x} - \mathbf{x}''|})$$

where $\mathbf{x} = (x, y)$, $\mathbf{x}' = (\xi, \eta)$, and $\mathbf{x}'' = (\xi, -\eta)$. But with the second piece of boundary, $\{x = 0\}$, we must reflect about that boundary also: $\mathbf{x}''' = (-\xi, \eta)$

(see Figure 1). Now we have

$$\frac{1}{2\pi}\ln(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) - \frac{1}{2\pi}\ln(\frac{1}{|\mathbf{x}-\mathbf{x}''|}) - \frac{1}{2\pi}\ln(\frac{1}{|\mathbf{x}-\mathbf{x}'''|}) \ .$$

The problem now is that if we have $\mathbf{x}$ on one boundary segment or the other segment, two of these log terms cancel out, but not three of them. To get $G = 0$ on both segments of the boundary of the quarter-plane Ω we need to add/subtract another term (that is harmonic in Ω). Pick the reflection point $\mathbf{x}^* = (-\xi, -\eta)$ and write

$$G(\mathbf{x}, \mathbf{x}') = G(x, y, \xi, \eta) =$$

$$\frac{1}{2\pi}\left\{\ln(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) - \ln(\frac{1}{|\mathbf{x}-\mathbf{x}''|}) - \ln(\frac{1}{|\mathbf{x}-\mathbf{x}'''|}) + \ln(\frac{1}{|\mathbf{x}-\mathbf{x}^*|})\right\} =$$

$$\frac{1}{2\pi}\left\{\ln(1/R) - \ln(1/R'') - \ln(1/R''') + \ln(1/R^*)\right\} \ .$$

It should be clear from Figure 1 that when $\mathbf{x} = (0, y)$, then $R''' = R$, $R^* = R''$, so $G_{|_{x=0}} = 0$. Similarly, when $\mathbf{x} = (x, 0)$, then $R'' = R$ and $R^* = R'''$, so $G_{|_{y=0}} = 0$. Since all the terms combined are harmonic in Ω, except when $(x, y) = (\xi, \eta)$, then G is the Green's function for the quarter-plane that we seek.

Example 4: Now consider the Poisson problem in the unit disk:

$$\begin{cases} \nabla^2 u = -f(r,\theta) & \text{in } \Omega = \{(r,\theta) : 0 \le r < 1, 0 \le \theta < 2\pi\} \\ u = 0 & \text{on } \partial\Omega = \{(1,\theta) : 0 \le \theta < 2\pi\} \end{cases}$$

Again fix $\mathbf{x}' = (\xi, \eta) = (s, \phi) \in \Omega$. Since the boundary can be described by a single function ($r = 1$), we should be able to use a single reflection point. A straight forward calculation using $(x, y) = (r, \theta) = (r\cos(\theta), r\sin(\theta))$ and $(\xi, \eta) = (s, \phi) = (s\cos(\phi), s\sin(\phi))$ gives

$$|\mathbf{x}-\mathbf{x}'|^2 = (x-\xi)^2 + (y-\eta)^2 = r^2 + s^2 - 2rs\cos(\theta - \phi) \ .$$

Let the reflected point (outside of the unit disk Ω) be $\mathbf{x}'' = (1/s, \phi)$, which is on the same "ray" as $(\xi, \eta) = (s, \phi)$, then $|\mathbf{x}-\mathbf{x}''|^2 = r^2 + \frac{1}{s^2} - 2\frac{r}{s}\cos(\theta - \phi)$.

Then, when $r = 1$, we have

$$\frac{1}{2\pi}\left\{\ln(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) - \ln(\frac{1}{|\mathbf{x}-\mathbf{x}''|})\right\} =$$

$$\frac{1}{2\pi}\ln\left\{\sqrt{\frac{1-(2/s)\cos(\theta-\phi)+1/s^2}{1-2s\cos(\theta-\phi)+s^2}}\right\} =$$

$$\frac{1}{2\pi}\ln\left\{\sqrt{\frac{1-(2/s)\cos(\theta-\phi)+1/s^2}{s^2(1/s^2-(2/s)\cos(\theta-\phi)+1)}}\right\} =$$

$$\frac{1}{2\pi}\ln(1/s)$$

which is a constant. But a constant is trivially a harmonic function, so write

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') = G(r,\theta,s,\phi) &= \frac{1}{2\pi}\left\{\ln(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) - \ln(\frac{1}{|\mathbf{x}-\mathbf{x}''|}) - \ln(\frac{1}{s})\right\} \\ &= \frac{1}{2\pi}\ln\left\{\frac{s|\mathbf{x}-\mathbf{x}''|}{|\mathbf{x}-\mathbf{x}'|}\right\} \ . \end{aligned}$$

Thus, we have the solution

$$u(r,\theta) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \ln\left\{\frac{s^2r^2-2rs\cos(\theta-\phi)+1}{r^2-2rs\cos(\theta-\phi)+s^2}\right\} f(s,\phi)\ sds\ d\phi\ . \quad (3)$$

Remark: It is possible to solve Laplace's equation problem

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : 0 \le r < 1, 0 \le \theta < 2\pi\} \\ u(1,\theta) = h(\theta) & \text{for } 0 \le \theta < 2\pi \end{cases}$$

by means of the Green's function approach. In this case the solution is

$$u(r,\theta) = \int_0^{2\pi} \frac{\partial G}{\partial r}(r,\theta,1,\phi)h(\phi)\ d\phi\ \ ,$$

where G is given above. By a tedious calculation, one can obtain from the derivative of G that

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} h(\phi)\, d\phi \tag{4}$$

which is exactly Poisson's integral formula derived earlier.

Remark: Given the more general problem

$$\begin{cases} \nabla^2 u = -f(r,\theta) & \text{in } \Omega \\ u = h(\theta) & \text{on } \partial\Omega \end{cases}$$

where $\Omega =$ unit disk, the solution would just be the sum of the two solutions (3),(4).

Exercises:

1. Show that the Green's function is unique. (Hint: take the difference of two of them.)

2. Find the Green's function for the tilted half-space $\Omega = \{(x,y) : ax + by > 0\}$.

3. Find the Green's function for the unit square $\Omega = \{(x,y) : 0 < x < 1, 0 < y < 1\}$.

4. With $G(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ as the free space Green's function in $\mathbb{R}^3$, construct the Green's function for the half-space $\Omega = \{(x,y,z) : x, y \in \mathbb{R}, z > 0\}$.

5. For the unit sphere Ω in $\mathbb{R}^3$ there is an analogue of the reflection point given for the unit disk from our example 4. What would it be in the 3D case? Construct the Green's function and write out the solution for the problem defined in $\Omega = \{(x,y,z) : x^2+y^2+z^2 < 1\}$

$$\begin{cases} \nabla^2 u = -f(x,y,z) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Remark: Recall from calculus Green's second identity

$$\int_\Omega (u\nabla^2 G - G\nabla^2 u)\, d\mathbf{x} = \int_{\partial\Omega} (u\nabla G - g\nabla u)\cdot \nu ds$$

and again consider the Poisson problem

$$\begin{cases} \nabla^2 u = -f(x,y) & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{5}$$

For arbitrary $(\xi, \eta) \in \Omega$, we know that $G = G(x, y, \xi, \eta)$ is harmonic for all $(x, y) \in \Omega$, $(x, y) \neq (\xi, \eta)$. Another way to think of G is that it solves the problem

$$\begin{cases} \nabla^2 G = -\delta(x-\xi, y-\eta) & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

where the δ is the standard 2D version of the Dirac distribution (see Appendix D). Then substituting this into Green's second identity, where u solves (5) and G solves (6), we have, from the boundary conditions,

$$0 = \int_{\partial\Omega} (u\nabla \cdot \nu - G\nabla u \cdot \nu) ds = \int_{\Omega} (u\nabla^2 G - G\nabla^2 u) d\mathbf{x} =$$
$$\int_{\Omega} (-u(\xi,\eta)\delta(x-\xi, y-\eta) + G(x,y,\xi,\eta) f(\xi,\eta)) d\xi d\eta =$$
$$-u(x,y) + \int_{\Omega} G(x,y,\xi,\eta) f(\xi,\eta) d\xi d\eta \ .$$

That is, $u(x, y) = \int_\Omega G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta$.

Summary: Know how to obtain reflection points for a given problem and construct the Green's function by the reflection principle. Then know how to write the solution formula in terms of the Green's function, being careful to have the correct limits of integration.

Appendix: Solution of Poisson problem example, page 1

Problem:

$$\begin{cases} u_{xx} + u_{yy} = -f(x,y) & \text{in } \Omega = \{(x,y) : 0 < x < \pi, 0 < y < \pi\} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Using the finite Fourier transform $u(x, y) = \sum_{n=1}^{\infty} b_n(y) \sin(nx)$, and $f(x, y) = \sum_{n=1}^{\infty} f_n(y) \sin(nx)$, then, in the usual manner of finding Fourier coefficients (and the orthogonality of the set $\{\sin(nx)\}$ on domain $(0, \pi)$), $b_n(y) =$

$\frac{2}{\pi}\int_0^\pi u(\xi,y)\sin(n\xi)\ d\xi,\ f_n(y) = \frac{2}{\pi}\int_0^\pi f(\xi,y)\sin(n\xi)\ d\xi$. Substituting these series for u, f into the PDE, we obtain

$$\frac{d^2 b_n}{dy^2} - n^2 b_n = -f_n(y)\ ,\quad b_n(0) = b_n(\pi) = 0\ .$$

A fundamental set of solutions, for any n, for the homogeneous equation is $\cosh(ny), \sinh(ny)$; to determine a particular solution to the non-homogeneous equation by the variation-of-parameters method, let $w(y) = w_1(y)\sinh(ny) + w_2\cosh(ny)$. Carrying through the calculation of the method, we find that

$$w_1(y) = -\frac{1}{n}\int_0^y f_n(\eta)\cosh(n\eta)\ d\eta\ \ ,\ \ w_2(y) = \frac{1}{n}\int_0^y f_n(\eta)\sinh(n\eta)\ d\eta\ ,$$

so putting these back into $w(y)$, we have

$$w(y) = \frac{1}{n}\int_0^y f_n(\eta)\{\sinh(n\eta)\cosh(ny) - \cosh(n\eta)\sinh(ny)\}\ d\eta =$$
$$-\frac{1}{n}\int_0^y f_n(\eta)\sinh(n(y-\eta))\ d\eta$$

since $\sinh(n\eta)\cosh(ny) - \cosh(n\eta)\sinh(ny) = -\sinh(n(y-\eta))$. The general solution to the $b_n(y)$ equation is now

$$b_n(y) = A_n\cosh(ny) + B_n\sinh(ny) - \frac{1}{n}\int_0^y f_n(\eta)\sinh(n(y-\eta))\ d\eta\ .$$

Apply the boundary conditions:

$b_n(0) = A_n = 0$, and $b_n(\pi) = 0 = B_n\sinh(n\pi) - \frac{1}{n}\int_0^\pi f_n(\eta)\sinh(n(\pi-\eta))\ d\eta$.

Hence,

$$B_n = \frac{1}{n\sinh(n\pi)} \int_0^\pi f_n(\eta)\sinh(n(\pi-\eta))\;d\eta \Rightarrow$$

$$b_n(y) = \frac{\sinh(ny)}{n\sinh(n\pi)} \int_0^\pi f_n(\eta)\sinh(n(\pi-\eta))\;d\eta - \frac{1}{n}\int_0^y f_n(\eta)\sinh(n(y-\eta))\;d\eta =$$

$$\frac{1}{n\sinh(n\pi)} \left\{ \int_0^y f_n(\eta)[\sinh(ny)\sinh(n(\pi-\eta)) - \sinh(n(y-\eta))\sinh(n\pi)]\;d\eta + \right.$$

$$\left. \int_y^\pi f_n(\eta)\sinh(ny)\sinh(n(\pi-\eta))]\;d\eta \right\} .$$

Using the hyperbolic function addition formulas to expand $\sinh(ny)\sinh(n(\pi-\eta)) - \sinh(n(y-\eta))\sinh(n\pi)$, it becomes

$$\cosh(ny)\sinh(n\eta)\sinh(n\pi) - \cosh(n\pi)\sinh(n\eta)\sinh(ny) = \sinh(n\eta)\sinh(n(\pi-y)) ,$$

so

$$b_n(y) = \frac{1}{n\sinh(n\pi)} \left\{ \int_0^y f_n(\eta)\sinh(n\eta)\sinh(n(\pi-y))\;d\eta + \right.$$

$$\left. \int_y^\pi f_n(\eta)\sinh(ny)\sinh(n(\pi-\eta))\;d\eta \right\} .$$

Now substitute into this solution the expression for $f_n(\cdot)$ given above, giving us

$$b_n(y) = \frac{2}{n\pi\sinh(n\pi)} \left\{ \int_0^y \int_0^\pi f(\xi,\eta)\sin(n\xi)\sinh(n\eta)\sinh(n(\pi-y))\;d\xi d\eta + \right.$$

$$\left. \int_y^\pi \int_0^\pi f(\xi,\eta)\sin(n\xi)\sinh(ny)\sinh(n(\pi-\eta))\;d\xi d\eta \right\} .$$

Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} b_n(y) \sin(nx)$$

$$= \int_0^{\pi} \int_0^{y} \sum_{n=1}^{\infty} \frac{2\sin(nx)}{n\pi \sinh(n\pi)} f(\xi,\eta) \sin(n\xi) \sinh(n\eta) \sinh(n(\pi - y)) \, d\eta d\xi \, +$$

$$\int_0^{\pi} \int_y^{\pi} \sum_{n=1}^{\infty} \frac{2\sin(nx)}{n\pi \sinh(n\pi)} f(\xi,\eta) \sin(n\xi) \sinh(ny) \sinh(n(\pi - \eta)) \, d\eta d\xi$$

$$= \int_0^{\pi} \int_0^{y} f(\xi,\eta) \sum_{n=1}^{\infty} \frac{2\sinh(ny)\sinh(n\xi)}{n\pi \sinh(n\pi)} \sin(nx) \sin(n\xi) \, d\eta d\xi$$

$$+ \int_0^{\pi} \int_y^{\pi} \text{same except roles of } \eta \text{ and } y \text{ are switched } d\eta d\xi \; .$$

Defining G by

$$G(x,y,\xi,\eta) = \begin{cases} \sum_{n=1}^{\infty} \frac{2\sinh(n\eta)\sinh(n\xi)}{n\pi\sinh(n\pi)} \sin(nx)\sin(n\xi) & \text{if } \eta \leq y \\ \sum_{n=1}^{\infty} \text{same series with roles of } \eta \text{ and } y \text{ switched} & \text{if } y \leq \eta \end{cases}$$

then we can write u as

$$u(x,y) = \int_0^{\pi} \int_0^{\pi} G(x,y,\xi,\eta) f(\xi,\eta) \, d\xi d\eta \; .$$